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Integrability of the $\mathcal{N} = 2$ boundary sine-Gordon model

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Abstract
We construct a boundary Lagrangian for the $\mathcal{N} = 2$ supersymmetric sine-Gordon model which preserves (B-type) supersymmetry and integrability to all orders in the bulk coupling constant g. The supersymmetry constraint is expressed in terms of matrix factorizations.

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1. Introduction

Soon after the work of Ghoshal and Zamolodchikov in [1] on factorizable S-matrices and boundary states in integrable boundary models, Warner considered in [2] boundary theories which possess in addition a $\mathcal{N} = 2$ supersymmetry structure. The key point of his approach was to include fermionic fields solely defined on the boundaries as previously introduced in [1] to ensure unbroken supersymmetries in the boundary theory.

In a similar spirit, Nepomechie thereafter considered supersymmetric extensions of the sine-Gordon model. In [3, 4] he constructed boundary Lagrangians for the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases, establishing supersymmetry and integrability which beforehand were considered to be incompatible in the presence of boundaries. For a nice review of this development, see [5], and [8, 9] for further related results.

The authors of [9] especially provide an explanation for the appearance of fermionic boundary degrees of freedom following [10, 11] by using a perturbative CFT approach. See also [12] for a classification of admissible boundary conditions in this context.

Whereas the Lagrangian for the $\mathcal{N} = 1$ case in the treatment of [3] is exact, the Lagrangian from [4] ensures integrability only to first order in the bulk coupling constant g. Up to this order the boundary Lagrangian contains three free continuous parameters.

The main goal of this paper is to extend the discussion of [4] and to construct a boundary Lagrangian which preserves supersymmetry and integrability in the sense of a conserved higher spin quantity as in [4] to all orders in the bulk coupling constant. Our final result will contain up to phases and discrete choices only a single free boundary parameter. The additional parameters from [4] are fixed in our case by constraints of higher order in the bulk coupling constant.

While our main focus will be the (Lorentzian) $\mathcal{N} = 2$ sine-Gordon theory, the supersymmetry considerations in the first part are valid for arbitrary superpotentials.

One of our initial motivations to consider the present problem is the appearance of the sine-Gordon model as the worldsheet theory describing strings in a particular Maldacena–Maoz background from [13]. These backgrounds are particular pp-wave solutions preserving at least four spacetime supersymmetries. For a flat transverse space they are exact superstring solutions [14, 15] whose worldsheet theories in light-cone gauge are given by $\mathcal{N} = (2, 2)$ supersymmetric Landau–Ginzburg models [13].

Branes in these general backgrounds without the inclusion of boundary fields have been studied in [16]. Our results give rise to additional supersymmetric brane configurations. It would be interesting to obtain a more detailed understanding of these new branes, in particular from a spacetime point of view.

Boundary Lagrangians with fermionic excitations as used in [2, 4] appeared recently in a different string theoretical context in [21, 22]. These authors gave a realization of a suggestion by Kontsevich to characterize B-type branes in particular Landau–Ginzburg models related to superconformal field theories in terms of matrix factorizations. This proposal was further studied in [23–31], see also [32] for a review and further references.

The boundary Lagrangian we will use for the sine-Gordon model is of the same type as those used in [21, 22]. It would be very interesting to see how to employ the methods of the previous references in the case of non-homogeneous worldsheet superpotentials to obtain further insight, for example, into the spectrum of the boundary theory.

Motivated by possible applications in string theory we define the boundary theories in this paper on a strip with topology $\mathbb{R} \times [0, \pi]$ instead of the half space as in [4]. The differences have only notational significance and do not affect any conclusions.

A possibly more serious difference compared to [4] derives from our choice of a Lorentzian worldsheet signature compared to the Euclidean setting in [4]. The structure of the bosonic fields is almost unaffected by this. Deviating reality properties of the fermions, however, make a direct comparison of the fermionic sectors subtle and we do not attempt to relate them via a Wick rotation.

Although the indefinite worldsheet metric does not directly simplify the calculations concerning the integrability, its consequences in particular in the fermionic sector make reality requirements more transparent than in [4]. This will be especially helpful for studying the structure of the boundary potential $B(z, \overline{z})$ and the conserved supersymmetries.

The paper is organized as follows. In the subsequent section 2 we will write down a boundary Lagrangian including fermionic boundary excitations and derive the resulting boundary conditions. In section 3 we will find conditions under which the boundary theory is $\mathcal{N} = 2$ supersymmetric. The relevant requirement will be the matrix factorization constraint from [21, 22].

In the final section 4 the integrability of the boundary theory for the particular sine-Gordon case will be considered. Additional information such as the explicit component form of the higher spin conserved currents from [7, 4], are supplied in the appendix.

2. Landau–Ginzburg models and boundary fermions

In the first part we will consider general $\mathcal{N} = 2$ supersymmetric Landau–Ginzburg models with flat target space and vanishing holomorphic Killing vector term. On worldsheets without boundaries these theories are described by the (component) Lagrangian

$$\mathcal{L}_{\text{bulk}} = \tfrac{1}{2}g_{j\bar{J}}\big(\partial_+ z^j \partial_- \bar{z}^{\bar{J}} + \partial_+ \bar{z}^{\bar{J}}\partial_- z^j + \mathrm{i}\overline{\psi}_+^{\bar{J}} \overset{\leftrightarrow}{\partial}_- \psi_+^j + \mathrm{i}\overline{\psi}_-^{\bar{J}} \overset{\leftrightarrow}{\partial}_+ \psi_-^j\big) \\ - \tfrac{1}{2}\partial_i\partial_j W(\mathbf{z})\psi_+^i\psi_-^j - \tfrac{1}{2}\partial_{\bar{i}}\partial_{\bar{J}}\overline{W}(\overline{\mathbf{z}})\overline{\psi}_-^{\bar{i}}\overline{\psi}_+^{\bar{J}} - \tfrac{1}{4}g^{i\bar{J}}\partial_i W(\mathbf{z})\partial_{\bar{J}}\overline{W}(\overline{\mathbf{z}}), \tag{1}$$

with $\partial_\pm = \partial_t \pm \partial_\sigma$, compare for example with [17–19].

When defined on a manifold with boundaries, one might either enforce boundary conditions in addition to the equations of motion from (1) or include boundary terms containing in particular fermionic boundary degrees of freedom and work with the resulting (boundary) equations of motion. To the best of our knowledge, the second approach goes back to the study of integrals of motion and factorizable S-matrices of integrable boundary theories in the seminal paper [1]. There the authors considered in particular the massive Ising model with boundary fermions and the bosonic sine-Gordon theory with an additional bosonic boundary potential.

Comparable boundary Lagrangians have thereafter been adopted in different ways in [2–4] in the context of supersymmetric integrable boundary field theories. For a construction of the boundary structure relying on a boundary quantum group, see [9, 10].

As mentioned in the introduction, the approach following [2, 4] has also recently appeared in a string theory context; compare for example with [21, 22].

In this paper we use a boundary Lagrangian following [4], see also [21, 22], given by

$$\mathcal{L}^{\sigma=\pi}_{\text{boundary}} = \frac{\mathrm{i}}{2}(b\psi_-\overline{\psi}_+ - b^*\psi_+\overline{\psi}_-) - \frac{\mathrm{i}}{2}a \overset{\leftrightarrow}{\partial}_t \bar{a} + B(z,\bar{z}) \\ + \frac{\mathrm{i}}{2}(\overline{F}'(\bar{z})\bar{a} + \overline{G}'(\bar{z})a)(\overline{\psi}_+ + \mathrm{e}^{\mathrm{i}\beta}\overline{\psi}_-) + \frac{\mathrm{i}}{2}(G'(z)\bar{a} + F'(z)a)(\psi_+ + \mathrm{e}^{-\mathrm{i}\beta}\psi_-). \tag{2}$$

As we will later on restrict attention to superpotentials depending on a single holomorphic coordinate, the boundary Lagrangian is written down containing only contributions along the $z = z^1$ direction at $\sigma = \pi$. The Lagrangian (2) is chosen to be manifestly real and the constant b is determined by consistency of the resulting (fermionic) boundary conditions to $b = \mathrm{e}^{-\mathrm{i}\beta}$; compare for example with [6].

2.1. The boundary conditions

In this part we will determine the boundary conditions resulting from the boundary Lagrangian (2). They are obtained from the variation of $\mathcal{L}_{\text{boundary}}$ together with boundary terms from partially integrated contributions in $\delta\mathcal{L}_{\text{bulk}}$.

The bosonic part of the bulk contributions from (1) is given by

$$-g_{i\bar{\imath}}(\delta z^i\partial_\sigma\bar{z}^{\bar{\imath}} + \delta\bar{z}^{\bar{\imath}}\partial_\sigma z^i)\big|_{\sigma=0}^{\pi} = -2\delta x^I\partial_\sigma x^I\big|_{\sigma=0}^{\pi}, \tag{3}$$

whereas the fermionic kinetic parts lead to

$$\tfrac{1}{2}g_{i\bar{\imath}}\big(-\mathrm{i}\overline{\psi}_+^{\bar{\imath}}\delta\psi_+^i + \mathrm{i}\delta\overline{\psi}_+^{\bar{\imath}}\psi_+^i + \mathrm{i}\overline{\psi}_-^{\bar{\imath}}\delta\psi_-^i - \mathrm{i}\delta\overline{\psi}_-^{\bar{\imath}}\psi_-^i\big)\big|_{\sigma=0}^{\pi} = \mathrm{i}\big(\psi_-^I\delta\psi_-^I - \psi_+^I\delta\psi_+^I\big)\big|_{\sigma=0}^{\pi}. \tag{4}$$

Altogether the boundary conditions for the $z = z^1$ direction at $\sigma = \pi$ are with these terms found to be

$$\partial_\sigma z = \partial_{\bar{z}}B(z,\bar{z}) + \frac{\mathrm{i}}{2}(\overline{F}''(\bar{z})\bar{a} + \overline{G}''(\bar{z})a)(\overline{\psi}_+ + \mathrm{e}^{\mathrm{i}\beta}\overline{\psi}_-) \tag{5}$$

$$\partial_t a = \tfrac{1}{2}\overline{F}'(\bar{z})(\overline{\psi}_+ + \mathrm{e}^{\mathrm{i}\beta}\overline{\psi}_-) + \tfrac{1}{2}G'(z)(\psi_+ + \mathrm{e}^{-\mathrm{i}\beta}\psi_-) \tag{6}$$

$$\psi_+ - \mathrm{e}^{-\mathrm{i}\beta}\psi_- = \overline{F}'(\bar{z})\bar{a} + \overline{G}'(\bar{z})a \tag{7}$$

together with the complex conjugates

$$\partial_\sigma \bar{z} = \partial_z B(z, \bar{z}) + \frac{i}{2}(G''(z)\bar{a} + F''(z)a)(\psi_+ + e^{-i\beta}\psi_-) \tag{8}$$

$$\partial_t \bar{a} = \tfrac{1}{2}\overline{G}'(\bar{z})(\overline{\psi}_+ + e^{i\beta}\overline{\psi}_-) + \tfrac{1}{2}F'(z)(\psi_+ + e^{-i\beta}\psi_-) \tag{9}$$

$$\overline{\psi}_+ - e^{i\beta}\overline{\psi}_- = G'(z)\bar{a} + F'(z)a. \tag{10}$$

Setting

$$A(z) = G'(z)\bar{a} + F'(z)a \tag{11}$$

$$\overline{A}(z) = \overline{F}'(\bar{z})\bar{a} + \overline{G}'(\bar{z})a \tag{12}$$

and using the suitable fermionic combinations

$$\begin{aligned} \theta_+ &= \tfrac{1}{2}(\psi_+ + e^{-i\beta}\psi_-) \qquad & \bar{\theta}_+ &= \tfrac{1}{2}(\overline{\psi}_+ + e^{i\beta}\overline{\psi}_-) \\ \theta_- &= \tfrac{1}{2}(\psi_+ - e^{-i\beta}\psi_-) \qquad & \bar{\theta}_- &= \tfrac{1}{2}(\overline{\psi}_+ - e^{i\beta}\overline{\psi}_-) \end{aligned} \tag{13}$$

$$\begin{aligned} \psi_+ &= \theta_+ + \theta_- \qquad & \psi_- &= e^{i\beta}(\theta_+ - \theta_-) \\ \overline{\psi}_+ &= \bar{\theta}_+ + \bar{\theta}_- \qquad & \overline{\psi}_- &= e^{-i\beta}(\bar{\theta}_+ - \bar{\theta}_-) \end{aligned} \tag{14}$$

the boundary conditions finally become

$$\partial_\sigma z = \partial_{\bar{z}} B(z, \bar{z}) + i\overline{A}'(\bar{z})\bar{\theta}_+ \tag{15}$$

$$\partial_t a = \overline{F}'(\bar{z})\bar{\theta}_+ + G'(z)\theta_+ \tag{16}$$

$$\theta_- = \tfrac{1}{2}\overline{A}(\bar{z}) \tag{17}$$

and

$$\partial_\sigma \bar{z} = \partial_z B(z, \bar{z}) + iA'(z)\theta_+ \tag{18}$$

$$\partial_t \bar{a} = \overline{G}'\bar{\theta}_+ + F'(z)\theta_+ \tag{19}$$

$$\bar{\theta}_- = \tfrac{1}{2}A(z). \tag{20}$$

By eliminating the fermionic boundary degrees of freedom in favour of $\theta_-, \bar{\theta}_-$ in (15) and (18), these bosonic boundary conditions take on the structure with a quadratic fermionic correction term as already discussed in [20] from a different point of view.

In the next section we will discuss how the so far undetermined holomorphic functions F, G are related to the superpotential if the boundary theory is to preserve B-type supersymmetries.

3. Matrix factorization and $\mathcal{N} = 2$ supersymmetry

The Landau–Ginzburg bulk theory of (1) has the four conserved supercurrents [18, 19]

$$G^0_\pm = g_{i\bar{j}}\partial_\pm \bar{z}^{\bar{j}}\psi^i_\pm \mp \frac{i}{2}\overline{\psi}^{\bar{j}}_\mp \partial_{\bar{j}}\overline{W} \qquad G^1_\pm = \mp g_{i\bar{j}}\partial_\pm \bar{z}^{\bar{j}}\psi^i_\pm - \frac{i}{2}\overline{\psi}^{\bar{j}}_\mp \partial_{\bar{j}}\overline{W} \tag{21}$$

$$\overline{G}^0_\pm = g_{i\bar{j}}\overline{\psi}^{\bar{j}}_\pm \partial_\pm z^i \pm \frac{i}{2}\psi^i_\mp \partial_i W \qquad \overline{G}^1_\pm = \mp g_{i\bar{j}}\overline{\psi}^{\bar{j}}_\pm \partial_\pm z^i + \frac{i}{2}\psi^i_\mp \partial_i W, \tag{22}$$

whose corresponding charges

$$Q_\pm = \int_0^{2\pi} d\sigma\, G^0_\pm, \qquad \overline{Q}_\pm = \int_0^{2\pi} d\sigma\, \overline{G}^0_\pm \tag{23}$$

represent the usual $\mathcal{N} = (2, 2)$ bulk supersymmetry.

As usual, the introduction of boundaries breaks at least a certain number of bulk symmetries. As explained in [18], there are essentially two possibilities of preserving a $\mathcal{N}=2$ supersymmetry algebra resulting from (21) and (22). Here we will concentrate on the so-called B type case.

Following [1, 18], the (B type) supersymmetries take on the general form

$$Q = \overline{Q}_+ + \mathrm{e}^{\mathrm{i}\beta}\overline{Q}_- + \Sigma_\pi(t) - \Sigma_0(t) \tag{24}$$

$$Q^\dagger = Q_+ + \mathrm{e}^{-\mathrm{i}\beta}Q_- + \overline{\Sigma}_\pi(t) - \overline{\Sigma}_0(t) \tag{25}$$

which includes (local) contributions of the boundary fields at $\sigma=\pi$ and $\sigma=0$. Using the conservation of the bulk fluxes (21) and (22), the boundary supersymmetries Q, $Q^\dagger$ are time independent, that is, conserved, if the fluxes fulfil the equations

$$0 = \overline{G}^1_+ + \mathrm{e}^{\mathrm{i}\beta}\overline{G}^1_-\Big|_{\sigma=\pi} - \dot{\Sigma}_\pi(t) \tag{26}$$

$$0 = \overline{G}^1_+ + \mathrm{e}^{\mathrm{i}\beta}\overline{G}^1_-\Big|_{\sigma=0} - \dot{\Sigma}_0(t) \tag{27}$$

together with their corresponding complex conjugates.

The boundary field $\Sigma_\pi(t)$ ($\Sigma_0(t)$) is here required to depend only on the bulk fields and their time derivatives at time t evaluated at $\sigma=\pi$ ($\sigma=0$) and the boundary degrees of freedom $a(t)$ and $\overline{a}(t)$.

3.1. W-factorization

In this section, we will solve the equation (26) to obtain a condition for the boundary fields $F(z)$, $G(z)$ and the boundary potential $B(z,\overline{z})$ for $\mathcal{N}=2$ supersymmetric branes.

From (26) and (22) we obtain

$$\begin{aligned}\partial_t\Sigma_\pi(t) &\overset{!}{=} \overline{G}^1_+ + \mathrm{e}^{\mathrm{i}\beta}\overline{G}^1_- \\ &= -\overline{\psi}_+\partial_+ z + \mathrm{e}^{\mathrm{i}\beta}\overline{\psi}_-\partial_- z + \frac{\mathrm{i}}{2}(\psi_- + \mathrm{e}^{\mathrm{i}\beta}\psi_+)\partial_z W\end{aligned} \tag{28}$$

evaluated at $\sigma=\pi$. Upon partial integration (28) leads to

$$\begin{aligned}\partial_\tau\Sigma_\pi(t) &= -\partial_t z(\overline{\psi}_+ - \mathrm{e}^{\mathrm{i}\beta}\overline{\psi}_-) - \partial_\sigma z(\overline{\psi}_+ + \mathrm{e}^{\mathrm{i}\beta}\overline{\psi}_-) + \frac{\mathrm{i}}{2}\partial_z W(z)(\psi_- + \mathrm{e}^{\mathrm{i}\beta}\psi_+) \\ &= -\partial_t(2z\overline{\theta}_-) + z((G''(z)\dot{z}\overline{a} + F''(z)\dot{z}a) + (G'(z)\dot{\overline{a}} + F'(z)\dot{a})) \\ &\quad - 2\overline{\theta}_+\partial_{\overline{z}}B(z,\overline{z}) + \mathrm{i}\,\mathrm{e}^{\mathrm{i}\beta}\theta_+\partial_z W(z)\end{aligned} \tag{29}$$

from which we get

$$\begin{aligned}\partial_t\Sigma_\pi(t) &= -\partial_t(2z\overline{\theta}_- - p(z)a - q(z)\overline{a}) + (zG''(z) - q'(z))\dot{z}\overline{a} + (zF''(z) - p'(z))\dot{z}a \\ &\quad + (zG'(z) - q(z))\dot{\overline{a}} + (zF'(z) - p(z))\dot{a} - 2\overline{\theta}_+\partial_{\overline{z}}B(z,\overline{z}) + \mathrm{i}\,\mathrm{e}^{\mathrm{i}\beta}\theta_+\partial_z W(z).\end{aligned} \tag{30}$$

Using

$$q'(z) = zG''(z) \quad\Rightarrow\quad q(z) = zG'(z) - G(z) \tag{31}$$

and

$$p'(z) = zF''(z) \quad\Rightarrow\quad p(z) = zF'(z) - F(z) \tag{32}$$

together with the equations of motion for a and $\overline{a}$ we finally arrive at

$$\begin{aligned}\partial_t\Sigma_\pi(t) &= -\partial_t(2z\overline{\theta}_- - p(z)a - q(z)\overline{a}) + \theta_+(G(z)F'(z) + F(z)G'(z) + \mathrm{i}\mathrm{e}^{\mathrm{i}\beta}\partial_z W(z)) \\ &\quad + 2\overline{\theta}_+\big(\tfrac{1}{2}G(z)\overline{G}'(\overline{z}) + \tfrac{1}{2}F(z)\overline{F}'(\overline{z}) - \partial_{\overline{z}}B(z,\overline{z})\big).\end{aligned} \tag{33}$$

The conditions for $\mathcal{N}=2$ supersymmetry therefore read

$$W(z) = \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\beta} F(z)G(z) + \mathrm{const} \tag{34}$$

$$B(z,\bar{z}) = \tfrac{1}{2}(F(z)\overline{F}(\bar{z}) + G(z)\overline{G}(\bar{z})) + \mathrm{const} \tag{35}$$

and the local boundary field Σ_π appearing in the 'boundary adjusted' supercharges (24) and (25) is given by

$$\Sigma_\pi = -2z\bar{\theta}_- + (zF' - F)a + (zG' - G)\bar{a}. \tag{36}$$

It explicitly contains contributions from the fermionic boundary degrees of freedom; compare for example with the results in [9].

The condition (34) is of course identical to the matrix factorization condition from [21, 22], whereas (35) so far only determines the structure of the boundary potential $B(z,\bar{z})$. It does not lead to a condition on F, G as the boundary potential remains functionally undetermined by the supersymmetry considerations.

In the context of matrix factorizations in string theory as in [21, 22] and the literature mentioned in the introduction the focus is on quasi-homogeneous superpotentials which lead in the infrared to a superconformal field theory. The latter requires a conserved $U(1)$ R-charge which should also be present in the boundary theory; compare for example with [31]. This additional condition requires factorizations of W into quasi-homogeneous functions.

It is worth pointing out that there is no corresponding restriction on F and G in our case. Integrability together with supersymmetry in the context of the boundary Lagrangian (2) will give particular trigonometric functions in the case of the sine-Gordon model, but supersymmetry on its own allows for more general choices.

Extending our treatment, one might, following [9], add purely bosonic boundary degrees of freedom to (2). This opens up the possibility for more general choices of F and G even when enforcing supersymmetry and integrability. We will not consider this possibility in this paper.

4. The $\mathcal{N}=2$ boundary sine-Gordon model

From now on we will specify the superpotential to

$$W(z) = -\mathrm{i}\lambda\cos(\omega z), \tag{37}$$

and restrict attention therefore to the $\mathcal{N}=2$ supersymmetric sine-Gordon model[1].

When defined on a manifold without boundaries this theory is well known to be a supersymmetric and integrable extension of the purely bosonic sine-Gordon theory [7]. Its first nontrivial conserved higher spin currents on whose conservation in the presence of a boundary we will concentrate in the following, were derived in [7, 4]. In our conventions they are given in appendix A.

By using (37) in (1), one can immediately derive the bulk equations of motion. They are given by

$$\partial_+\partial_- z = -\mathrm{i}g\sin\bar{z}\,\overline{\psi}_-\overline{\psi}_+ - g^2\sin z\cos\bar{z} \tag{38}$$

$$\partial_+\partial_-\bar{z} = \mathrm{i}g\sin z\,\psi_+\psi_- - g^2\cos z\sin\bar{z} \tag{39}$$

$$\partial_-\psi_+ = g\cos\bar{z}\,\overline{\psi}_- \tag{40}$$

[1] The phase accompanying the real coupling constant λ is chosen for later convenience. Its form does not affect purely bosonic terms in the Lagrangian (1). In the fermionic parts of (1) it can be absorbed in a redefinition $\psi_\pm \to \mathrm{e}^{\mathrm{i}\alpha}\psi_\pm$, $\overline{\psi}_\pm \to \mathrm{e}^{-\mathrm{i}\alpha}\overline{\psi}_\pm$.

$$\partial_- \overline{\psi}_+ = g \cos z \, \psi_- \tag{41}$$

$$\partial_+ \psi_- = -g \cos \overline{z} \, \overline{\psi}_+ \tag{42}$$

$$\partial_+ \overline{\psi}_- = -g \cos z \, \psi_+, \tag{43}$$

where we set $\omega = 1$ and redefined the bulk coupling constant to $g = \frac{\lambda}{2}$, resembling the choices in [4].

4.1. Integrability in the presence of a boundary

In this section we will consider the $\mathcal{N} = 2$ sine-Gordon model in the presence of a boundary and derive conditions under which the following 'energy-like' combination of the bulk conserved quantities

$$I_3 = \int_0^\pi \mathrm{d}\sigma \,(T_4 + \overline{T}_4 - \theta_2 - \overline{\theta}_2) - \Sigma_\pi^{(3)}(t) + \Sigma_0^{(3)}(t) \tag{44}$$

is conserved when using the boundary Lagrangian (2). The inclusion of local boundary currents as $\Sigma_0^{(3)}(t)$ and $\Sigma_\pi^{(3)}(t)$ goes back to [1]. Their appearance is by now a well-known and frequently used feature in the context of integrable boundary field theories. It is in particular independent of the in our case present supersymmetries. The conservation of a higher spin quantity like I_3 is usually regarded as providing strong evidence for the integrability of the underlying two dimensional (boundary) field theory.

As previously done in section 3 for the supercurrents (21), (22) in (24) and (25), the quantity I_3 is conserved if the condition

$$\partial_t \Sigma_\pi^{(3)} = T_4 - \overline{T}_4 + \theta_2 - \overline{\theta}_2 \tag{45}$$

holds at $\sigma = \pi$. In deriving (45) we have used the equations (A.5) and (A.6) from appendix A. As before, there is an identical equation at $\sigma = 0$.

Due to the complexity of the conserved currents as given in appendix A, the calculation transforming the right hand side of (45) to a total time derivative is rather lengthy and intricate. It nevertheless follows a straightforward strategy which in our case differs slightly from the approach in [4].

In a first step we use the equations of motion (38)–(43) and the bosonic boundary conditions (15) and (18) to remove all σ derivatives on the bosonic and fermionic fields appearing in $T_4, \overline{T}_4, \theta_2, \overline{\theta}_2$.

In a second step we remove (where possible) all time derivatives on the fermionic fields θ_+ and $\overline{\theta}_+$ by partial integration and apply the identities from appendix B to furthermore replace θ_- and $\overline{\theta}_-$ and their time derivatives by the fermionic boundary fields a and $\overline{a}$.

In doing so a large number of terms cancel manifestly. There are, however, other terms as for example those proportional to combinations like $(\theta_+ \partial_t \theta_+)$ or $(\theta_+ \overline{\theta}_+)$ which cannot be reduced further and which cannot be written as a time derivative of a local field. Their prefactors given by expressions containing the boundary potential $B(z, \overline{z})$ and the functions $F(z), G(z)$ and their derivatives therefore necessarily have to vanish.

Together with the conditions (35) and (34) for the $\mathcal{N} = 2$ supersymmetry these resulting differential equations actually will be seen to determine the boundary Lagrangian up to two possible choices for the boundary potential including a free parameter and two additional (discrete) choices in prefactors appearing in the functions F and G.

In the following we will write down the differential equations determined as explained above and give their solutions. The explicit form of the boundary field $\Sigma_\pi^{(3)}$ appearing in (44) will be provided in the appendix.

4.2. The boundary potential $B(z, \bar{z})$

As explained in [4], the boundary potential $B(z, \bar{z})$ is already determined from the purely bosonic terms (A.1), (A.3) and (A.3), (A.4). The differential equations for the *real* field B read

$$0 = \partial_z \partial_z \partial_{\bar{z}} B + \tfrac{1}{4} \partial_{\bar{z}} B \tag{46}$$

$$0 = \partial_{\bar{z}} \partial_{\bar{z}} \partial_z B + \tfrac{1}{4} \partial_z B \tag{47}$$

together with

$$\partial_z \partial_z B = \partial_{\bar{z}} \partial_{\bar{z}} B. \tag{48}$$

This determines B to

$$B(z, \bar{z}) = \alpha \cos \frac{z - z_0}{2} \cos \frac{\bar{z} - \bar{z}_0}{2} + b, \qquad \alpha, b \in \mathbb{R}, \quad z_0 \in \mathbb{C} \tag{49}$$

which is so far exactly the result of [4]. Together with (35) we will nevertheless find further conditions on the so far unspecified constant z_0 which come from contributions of higher order in the bulk coupling constant g than considered in [4].

4.3. The boundary functions F, G, and $\overline{F}$, $\overline{G}$

From terms quadratic in the fermionic degrees of freedom as for example from

$$16\mathrm{i}(\partial_t z)^3 \big(A'''(z) + \tfrac{1}{4} A'(z)\big)\theta_+ \tag{50}$$

and

$$48\mathrm{i}\, \partial_t z \partial_t{}^2 z \big(A''(z) + \tfrac{1}{4} A\big)\theta_+ \tag{51}$$

we obtain the differential equations

$$0 = A''(z) + \tfrac{1}{4} A(z) \tag{52}$$

$$0 = \partial_z [F'(z) G'(z)] + \tfrac{1}{2} g \sin z\, \mathrm{e}^{\mathrm{i}\beta} \tag{53}$$

together with the corresponding complex conjugates.

From (52) and (11), (12) the functions $F(z)$ and $G(z)$ are determined to

$$F(z) = A_0 \cos \frac{z - \kappa_1}{2} + C_0 \tag{54}$$

$$G(z) = B_0 \cos \frac{z - \kappa_2}{2} + D_0, \tag{55}$$

and the equation (53) becomes with the matrix factorization condition (34)

$$\begin{aligned} 0 &= F' \left(G'' + \frac{1}{4} G \right) + G' \left(F'' + \frac{1}{4} F \right) \\ &= -\frac{1}{8} \left(A_0 D_0 \sin \frac{z - \kappa_1}{2} + B_0 C_0 \sin \frac{z - \kappa_2}{2} \right). \end{aligned} \tag{56}$$

By combining these results with the expression for $B(z, \bar{z})$ found in (35), we can in the next step deduce conditions on the so far free parameters in (54), (55) and (49).

Using (54) and (55) in the differentiated condition (34), we obtain

$$2g\, \mathrm{e}^{\mathrm{i}\beta} \sin z = -\frac{A_0 B_0}{2} \cos \frac{\kappa_1 + \kappa_2}{2} \sin z + \frac{A_0 B_0}{2} \sin \frac{\kappa_1 + \kappa_2}{2} \cos z \tag{57}$$

and therefore

$$\kappa_1 + \kappa_2 = 2\pi n \qquad n \in \mathbb{Z} \tag{58}$$

$$2g\,\mathrm{e}^{\mathrm{i}\beta} = -\tfrac{1}{2}A_0 B_0 (-)^n. \tag{59}$$

A constraint on z_0 appearing in the boundary potential B comes from equation (35). In particular, we have

$$\partial_z \partial_{\bar{z}} B(z, \bar{z}) = \tfrac{1}{2}(F'\overline{F}' + G'\overline{G}'). \tag{60}$$

Using (49) and (54), (55) evaluated at $z = z_0$, we get

$$0 = A_0 \overline{A}_0 \sin\frac{z_0 - \kappa_1}{2} \sin\frac{\bar{z}_0 - \overline{\kappa_1}}{2} + B_0 \overline{B}_0 \sin\frac{z_0 - \kappa_2}{2} \sin\frac{\bar{z}_0 - \overline{\kappa_2}}{2} \tag{61}$$

and therefore

$$0 = \sin\frac{z_0 - \kappa_1}{2}, \qquad 0 = \sin\frac{z_0 - \kappa_2}{2}. \tag{62}$$

Together with (58) and the observation that the boundary Lagrangian (2) does not depend on the constants C_0, D_0 in (54) and (55), we therefore obtain the two following possibilities ensuring integrability in the sense discussed above.

Case I:

$$B(z, \bar{z}) = \alpha \cos\frac{z}{2} \cos\frac{\bar{z}}{2}, \qquad F(z) = A_0 \cos\frac{z}{2}, \qquad G(z) = B_0 \cos\frac{z}{2} \tag{63}$$

with

$$A_0 B_0 = -4g\,\mathrm{e}^{\mathrm{i}\beta}, \qquad A_0 \overline{A}_0 + B_0 \overline{B}_0 = 2\alpha \tag{64}$$

Case II:

$$B(z, \bar{z}) = \alpha \sin\frac{z}{2} \sin\frac{\bar{z}}{2}, \qquad F(z) = A_0 \sin\frac{z}{2}, \qquad G(z) = B_0 \sin\frac{z}{2} \tag{65}$$

with

$$A_0 B_0 = 4g\,\mathrm{e}^{\mathrm{i}\beta}, \qquad A_0 \overline{A}_0 + B_0 \overline{B}_0 = 2\alpha. \tag{66}$$

From (64) and (66) we have, in both cases,

$$A_0 \overline{A}_0 = \alpha \pm \sqrt{\alpha^2 - 16g^2} \tag{67}$$

and therefore

$$A_0^{\pm} = \mathrm{e}^{\mathrm{i}\gamma} \sqrt{\alpha \pm \sqrt{\alpha^2 - 16g^2}}. \tag{68}$$

The undetermined phase γ appearing in (67) can be absorbed in a redefinition of the fermionic boundary fields a and $\bar{a}$. From (67) we furthermore have the condition

$$\alpha \geqslant 4g \geqslant 0 \tag{69}$$

which in particular leads to a positive, semidefinite boundary potential $B(z, \bar{z})$ in (2).

With these choices all remaining terms in (45) either vanish or can be written as a total time derivative as a rather long calculation shows. This ensures therefore the conservation of the higher spin quantity (44) in the presence of a boundary to all orders in the bulk coupling constant g, providing strong evidence for integrability.

To first order in g, the condition (60) together with (66) and (67) does not give a constraint on F and G and one reobtains the situation of [4] where the two additional (real) parameters expressed by z_0 of the boundary potential in (49) were found to be compatible with integrability to that order.

5. Conclusions

Following [2, 4, 21, 22], we constructed boundary Lagrangians which establish supersymmetry and integrability in the sense of a conserved higher spin current for the (Lorentzian) $\mathcal{N} = 2$ supersymmetric sine-Gordon model defined on the strip or the half space.

In contradistinction to [4], our Lagrangians are exact to all orders in the bulk coupling constant g. Apart from phases and the bulk coupling constant g, both possible choices contain a single continuous parameter α. The additional free complex parameter z_0 appearing in [4] is in our case essentially fixed to $z_0 = 0$ or $z_0 = \pi$.

The ansatz (2) for the boundary Lagrangian secures supersymmetry for a general superpotential if the boundary functions F, G fulfil the matrix factorization condition (34). Following [9], one might, furthermore, add purely bosonic boundary degrees of freedom to (2). It would be interesting to study this situation in greater detail.

As mentioned in the introduction, one can readily apply the present setup to the construction of branes in Maldacena–Maoz backgrounds [13] leading to branes that generalize the constructions of [16]. This will be considered elsewhere.

Apart from the construction of further branes in Maldacena–Maoz backgrounds it would be interesting to obtain a more detailed (spacetime) interpretation of the boundary fermions and the corresponding branes. At least for the integrable sine-Gordon case it might furthermore be possible to derive detailed information about the string spectrum by using methods like the thermodynamic Bethe Ansatz, see for example [33–37], and [38] for a related setting.

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Appendix A. Higher spin conserved currents

Using the superfield approach from [7], the first higher spin conserved bulk currents for the $\mathcal{N} = 2$ supersymmetric sine-Gordon model were determined in components by Nepomechie in [4] to (in our conventions)

$$\begin{aligned} T_4 = {} & -(\partial_+\bar{z})^3\partial_+ z - (\partial_+ z)^3\partial_+\bar{z} + 2\partial_+^2\bar{z}\partial_+^2 z - \mathrm{i}\overline{\psi}_+\partial_+\psi_+[(\partial_+\bar{z})^2 + 3(\partial_+ z)^2] \\ & - 2\mathrm{i}\psi_+\partial_+\overline{\psi}_+(\partial_+\bar{z})^2 - 2\mathrm{i}\overline{\psi}_+\psi_+\partial_+\bar{z}\partial_+^2\bar{z} + 2\mathrm{i}\partial_+\overline{\psi}_+\partial_+^2\psi_+ \end{aligned} \tag{A.1}$$

$$\begin{aligned} \theta_2 = {} & g^2\sin z\sin\bar{z}[(\partial_+\bar{z})^2 + (\partial_+ z)^2] - 2g^2\cos z\cos\bar{z}\partial_+ z\partial_+\bar{z} - \mathrm{i}g\cos z\psi_-\psi_+[(\partial_+\bar{z})^2 + (\partial_+ z)^2] \\ & - 2\mathrm{i}g\sin z\partial_+ z[-\psi_+\partial_+\psi_- + \psi_-\partial_+\psi_+] + 2\mathrm{i}g\cos z\partial_+\psi_-\partial_+\psi_+ \\ = {} & g^2\sin z\sin\bar{z}[(\partial_+\bar{z})^2 + (\partial_+ z)^2] - 2g^2\cos z\cos\bar{z}\partial_+ z\partial_+\bar{z} \\ & - 2\mathrm{i}g^2\sin z\cos\bar{z}\partial_+ z\psi_+\overline{\psi}_+ - 2\mathrm{i}g^2\cos z\cos\bar{z}\overline{\psi}_+\partial_+\psi_+ \\ & - \mathrm{i}g\cos z\psi_-\psi_+[(\partial_+\bar{z})^2 + (\partial_+ z)^2] - 2\mathrm{i}g\sin z\partial_+ z\psi_-\partial_+\psi_+ \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} \overline{T}_4 = {} & -(\partial_-\bar{z})^3\partial_- z - (\partial_- z)^3\partial_-\bar{z} + 2\partial_-^2\bar{z}\partial_-^2 z - \mathrm{i}\overline{\psi}_-\partial_-\psi_-[(\partial_-\bar{z})^2 + 3(\partial_- z)^2] \\ & - 2\mathrm{i}\psi_-\partial_-\overline{\psi}_-(\partial_-\bar{z})^2 - 2\mathrm{i}\overline{\psi}_-\psi_-\partial_-\bar{z}\partial_-^2\bar{z} + 2\mathrm{i}\partial_-\overline{\psi}_-\partial_-^2\psi_- \end{aligned} \tag{A.3}$$

$$\begin{aligned}\overline{\theta}_2 = {} & g^2 \sin z \sin \overline{z}[(\partial_- \overline{z})^2 + (\partial_- z)^2] - 2g^2 \cos z \cos \overline{z}\partial_- z\partial_- \overline{z} \\ & - \mathrm{i}g \cos z\psi_-\psi_+[(\partial_- \overline{z})^2 + (\partial_- z)^2] \\ & - 2\mathrm{i}g \sin z\partial_- z(-\psi_+\partial_-\psi_- + \psi_-\partial_-\psi_+) + 2\mathrm{i}g \cos z\partial_-\psi_-\partial_-\psi_+ \\ = {} & g^2 \sin z \sin \overline{z}[(\partial_- \overline{z})^2 + (\partial_- z)^2] - 2g^2 \cos z \cos \overline{z}\partial_- z\partial_- \overline{z} \\ & - 2\mathrm{i}g^2 \sin z \cos \overline{z}\partial_- z\psi_-\overline{\psi}_- + 2\mathrm{i}g^2 \cos z \cos \overline{z}\partial_-\psi_-\overline{\psi}_- \\ & - \mathrm{i}g \cos z\psi_-\psi_+[(\partial_- \overline{z})^2 + (\partial_- z)^2] + 2\mathrm{i}g \sin z\partial_- z\psi_+\partial_-\psi_-. \end{aligned} \tag{A.4}$$

By using the equations of motion the currents fulfil

$$\partial_- T_4 = \partial_+\theta_2 \tag{A.5}$$

$$\partial_+\overline{T}_4 = \partial_-\overline{\theta}_2, \tag{A.6}$$

and are therefore leading to conserved spin 3 quantities in the bulk theory.

Appendix B. $A(z)$, $\overline{A}(\overline{z})$-identities

Using the boundary conditions (15)–(20) we have the following identities at $\sigma = \pi$ used in the boundary expansion of the bulk conserved currents $T_4, \overline{T}_4$ and $\theta_2, \overline{\theta}_2$

$$\partial_t\theta_- = \tfrac{1}{2}\partial_t\overline{A}(\overline{z}) = \tfrac{1}{2}\overline{A}'(\overline{z})\partial_t \overline{z} + \overline{F}'\overline{G}'\overline{\theta}_+ + \partial_z\partial_{\overline{z}}B\theta_+ \tag{B.1}$$

$$\partial_t\overline{\theta}_- = \tfrac{1}{2}\partial_t A(z) = \tfrac{1}{2}A'(z)\partial_t z + \partial_z\partial_{\overline{z}}B\overline{\theta}_+ + F'G'\theta_+ \tag{B.2}$$

$$\partial_t A'(z) = A''(z)\partial_t z + 2\partial_z\partial_z\partial_{\overline{z}}B\overline{\theta}_+ + \partial_z(G'F')\theta_+ \tag{B.3}$$

$$\partial_t\overline{A}'(\overline{z}) = \overline{A}''(\overline{z})\partial_t \overline{z} + \partial_{\overline{z}}(\overline{F}'\overline{G}')\overline{\theta}_+ + 2\partial_z\partial_{\overline{z}}\partial_{\overline{z}}B\theta_+ \tag{B.4}$$

and

$$\partial_t A''(z) = A'''(z)\partial_t z + (G'''\overline{G}' + F'''\overline{F}')\overline{\theta}_+ + (G'''F' + F'''G')\theta_+ \tag{B.5}$$

$$\partial_t\overline{A}''(\overline{z}) = \overline{A}'''(\overline{z})\partial_t \overline{z} + (\overline{F}'''\overline{G}' + \overline{G}'''\overline{F}')\overline{\theta}_+ + (\overline{F}'''F' + \overline{G}'''G')\theta_+. \tag{B.6}$$

Quadratic fermionic terms as $A(z)\overline{A}(\overline{z})$, furthermore, lead to identities like

$$A(z)\overline{A}(\overline{z}) = (G'(z)\overline{G}' - F'(z)\overline{F}'(\overline{z}))\overline{a}a \tag{B.7}$$

$$A(z)A'(z) = (G'(z)F''(z) - F'(z)G''(z))\overline{a}a. \tag{B.8}$$

Appendix C. The boundary current $\Sigma_\pi^{(3)}(t)$

In this appendix we present the explicit form of the local boundary term $\Sigma_\pi^{(3)}(t)$ appearing in the conserved quantity (44). It is given by

$$\begin{aligned}\Sigma_\pi^{(3)}(t) = {} & 16\mathrm{i}\partial_t{}^2 zA'(z)\theta_+ + 16\mathrm{i}\partial_t^2 \overline{z}\overline{A}'(\overline{z})\overline{\theta}_+ + 8\partial_z\partial_z B(\partial_t z)^2 + 8\partial_{\overline{z}}\partial_{\overline{z}}B(\partial_t \overline{z})^2 + 16\partial_z\partial_{\overline{z}}B\partial_t \overline{z}\partial_t z \\ & + 8\mathrm{i}g^2 \sin z \cos \overline{z}A'(z)\theta_+ + 8\mathrm{i}g^2 \sin \overline{z} \cos z\overline{A}'(\overline{z})\overline{\theta}_+ - 4\mathrm{i}\overline{\theta}_-\theta_+((\partial_z B)^2 \\ & + 3(\partial_{\overline{z}}B)^2) - 4\mathrm{i}\overline{\theta}_-\theta_+((\partial_t \overline{z})^2 + 3(\partial_t z)^2) - 8\mathrm{i}\theta_-\overline{\theta}_+((\partial_t \overline{z})^2 + (\partial_z B)^2) \\ & - 16\mathrm{i}g \cos z\, \mathrm{e}^{\mathrm{i}\beta}(\theta_+\partial_t\theta_+ - \theta_-\partial_t\theta_-) + 32\mathrm{i}\partial_t\overline{\theta}_-\partial_t\theta_+ + 16\mathrm{i}g^2 \cos z \cos \overline{z}\overline{\theta}_+\theta_- \\ & - 8\mathrm{i}g \sin \overline{z}\, \mathrm{e}^{-\mathrm{i}\beta}\partial_z B\overline{\theta}_+\overline{\theta}_- + 16\mathrm{i}\overline{\theta}_+\overline{A}'(\overline{z})\partial_t{}^2 \overline{z} - 16\mathrm{i}A'(z)\partial_t{}^2 z\theta_+ \\ & + 32\mathrm{i}\partial_z\partial_z\partial_{\overline{z}}B\partial_t z\overline{\theta}_+\theta_+ + 16\mathrm{i}(\partial_t \overline{z})^2\overline{\theta}_+\overline{A}''(\overline{z}) - 16\mathrm{i}(\partial_t z)^2 A''(z)\theta_+ \\ & - 8\mathrm{i}g^2 \cos z \cos \overline{z}\overline{\theta}_-\theta_+ + 8\mathrm{i}g \sin z\, \mathrm{e}^{\mathrm{i}\beta}\partial_{\overline{z}}B\theta_-\theta_+ - 8\mathrm{i}\partial_z B\partial_t \overline{z}\overline{\theta}_-\theta_- \\ & + 16\partial_z BA'(z)\theta_-\overline{\theta}_+\theta_+ + H_1(z, \overline{z}) + H_2(z, \overline{z}) \end{aligned} \tag{C.1}$$

with

$$\partial_z H_1(z,\bar{z}) = (-2(\partial_z B)^3 - 6\partial_z B(\partial_{\bar{z}} B)^2) \tag{C.2}$$

$$\partial_{\bar{z}} H_1(z,\bar{z}) = (-2(\partial_{\bar{z}} B)^3 - 6\partial_{\bar{z}} B(\partial_z B)^2) \tag{C.3}$$

and

$$\begin{aligned}\partial_z H_2(z,\bar{z}) = +4g^2[&\sin z \sin\bar{z}\partial_{\bar{z}} B - \cos z\cos\bar{z}\partial_z B \\ &+ 2\sin z\cos\bar{z}\partial_z\partial_z B + 2\cos z\sin\bar{z}\partial_z\partial_{\bar{z}} B]\end{aligned} \tag{C.4}$$

$$\begin{aligned}\partial_{\bar{z}} H_2(z,\bar{z}) = +4g^2[&\sin z \sin\bar{z}\partial_z B - \cos z\cos\bar{z}\partial_{\bar{z}} B \\ &+ 2\sin z\cos\bar{z}\partial_z\partial_{\bar{z}} B + 2\cos z\sin\bar{z}\partial_{\bar{z}}\partial_{\bar{z}} B].\end{aligned} \tag{C.5}$$

For the choice $B(z,\bar{z}) = \alpha\sin\frac{z}{2}\sin\frac{\bar{z}}{2}$, these functions read

$$H_1(z,\bar{z}) = \tfrac{1}{12}\alpha^2(-4 + \cos z + \cos\bar{z} + 2\cos z\cos\bar{z})B(z,\bar{z}) \tag{C.6}$$

$$H_2(z,\bar{z}) = 4g^2 B(z,\bar{z}). \tag{C.7}$$

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